THE SIGNIFICANCE OF WEIERSTRASS'S THEOREM.

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1. Introduction. Certain theorems that are commonly presented in mathematical instruction appear at first sight to be of theoretical importance only. Actual application of some such theorems seems very far fetched, and it is often not sought for because the possibility of its realization seems slight.

Among such theorems is that of Weierstrass on polynomial approximations to a continuous function:*

Given a function $f(x)$ continuous in the interval $a \gtreqless x \gtreqless b$, and a positive number ϵ, then a polynomial $P(x)$ exists, such that

$$|f(x) - P(x)| < \epsilon, \qquad a \gtreqless x \gtreqless b.$$

It is the purpose of this paper to show how this theorem can be clarified to students, and how the contrast between the approximations that it expresses and those given by Taylor's series can be illuminated.

2. Contrast with Taylor Approximating Polynomials. The characteristic property of the successive polynomial approximations

$$(1) \qquad T_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

that are given by a Taylor series, is that *the coefficient a_i of a given power x^i is the same for every approximation past the first in which it occurs.*

The polynomials $P(x)$ of Weierstrass's theorem do not in general have this property, even when $f(x)$ is analytic. If $f(x)$ is not analytic, it is impossible that they should have that property.

To illustrate the manner in which polynomial approximations may arise that do not have the Taylor property, let us consider, for example, the amount of stretching in a steel wire due to variable tension. Hooke's law is expressed by the formula

$$(2) \qquad y = y_0(1 + kx),$$

where y_0 is the original length of the wire, and y is its length under a tension x. The constant k is determined by experiment upon a given wire.

It is known that Hooke's law is only approximately correct. To take into account minor variation not expressed by that law, we may assume a formula of the type

$$(3) \qquad y = y_0(1 + k_1x + k_2x^2),$$

and determine the constants k_1 and k_2 by a series of experiments. If this be done, *the coefficient k_1 of x in* (3) *may or may not, logically, coincide with the value of k in* (2). If it does, it manifests the characteristic Taylor property; if not, the polynomial approximations (2) and (3) are typical rather of the Weierstrass behavior. As a matter of fact, as we shall see below, k_1 is *not* equal to k.

* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 422.

For still greater accuracy, or for theoretical purposes, we might insert still higher powers of x in the assumed formula for y:

$$y = y_0(1 + k_1^{(n)}x + k_2^{(n)}x^2 + \cdots + k_n^{(n)}x^n), \tag{4}$$

where $k_i^{(n)}$ denotes the coefficient of x^i in the nth one of these approximations. The question that arises is whether $k_i^{(n)}$ is or is not independent of n; if it is, the polynomials have the Taylor property mentioned above; if not, the coefficient of the same power of x varies from one approximation to another in the Weierstrass manner. We proceed to show that the latter alternative is the usual one.

3. Computation of Coefficients by Least Squares. In such instances as that mentioned in article 2, the coefficients k_i are usually computed by the method of least squares. If the relation between two observed quantities x and y is of the form (4), it is usual to select the values of the coefficients $k_i^{(n)}$ which render the sum of the squares of the differences between the two sides of (4) a minimum, the sum being extended over all the sets of observed values. This is equivalent to solving for $k_i^{(n)}$ the following set of linear equations:

$$\tag{5} \left\{\begin{array}{l} k_1^{(n)}\Sigma x_j^2 + k_2^{(n)}\Sigma x_j^3 + \cdots + k_n^{(n)}\Sigma x_j^{n+1} = \Sigma x_j\left(\dfrac{y_j - y_0}{y_0}\right), \\ k_1^{(n)}\Sigma x_j^3 + k_2^{(n)}\Sigma x_j^4 + \cdots + k_n^{(n)}\Sigma x_j^{n+2} = \Sigma x_j^2\left(\dfrac{y_j - y_0}{y_0}\right), \\ \quad\cdot \qquad\qquad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ \quad\cdot \qquad\qquad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ \quad\cdot \qquad\qquad \cdot \qquad\qquad \cdot \qquad\qquad \cdot \\ k_1^{(n)}\Sigma x_j^{n+1} + k_2^{(n)}\Sigma x_j^{n+2} + \cdots + k_n^{(n)}\Sigma x_j^{2n} = \Sigma x_j^n\left(\dfrac{y_j - y_0}{y_0}\right), \end{array}\right.$$

where x_j and y_j are a pair of corresponding observed values of x and y, and where the summations are extended over all the pairs of observed values.

It is evident from these formulas that the approximation of next higher degree will give a set of coefficients $k_i^{(n+1)}$ that are in general very different from $k_i^{(n)}$, since the inclusion of a term in x^{n+1} in (4) results in increasing the number of equations in (5), as well as in adding a term to each left-hand side. It follows that *the successive approximations given by the theory of least squares have the characteristic Weierstrass property, i. e., the coefficient $k_i^{(n)}$ of x^i depends on n.*

4. Failure of Taylor Series. It is well known that many continuous functions, and even many functions that possess all their derivatives, are not expansible in Taylor series. Thus the function e^{-1/x^2} possesses all its derivatives for every value of x; nevertheless it is not expansible in Taylor series. A function that fails to have a derivative—of no matter how high an order—for a given value of x, is not expansible in Taylor series about that point. It is a result of Weierstrass's theorem that such approximations as (4) above, are not affected by these failures of Taylor's series.

We have seen in Art. 3 that the approximations of the form (4) ought not to be considered as the first terms of a Taylor series. It now becomes evident

that it is not even desirable to do so if it were allowable. For there are serious limitations upon the applicability of Taylor's (infinite) series, whereas *any continuous function* may be treated by Weierstrass's theorem. Doubtless most scientific phenomena are continuous in the range of experimentation; they certainly are so to within the errors of observation.

It follows that the frequent occurence in science of polynomial approximations of the type (4) constitutes an illustration of Weierstrass's theorem rather than of Taylor's theorem. Quite contrary to the spirit of apologetic hesitation that is sometimes manifested by experimenters in using what are popularly known as "the first terms of a Taylor series" to represent an observed quantity, Weierstrass's theorem constitutes a complete justification for the use of formulas of the type (4) in any scientific experiment.

ON THE IMPOSSIBILITY OF CERTAIN DIOPHANTINE EQUATIONS AND SYSTEMS OF EQUATIONS.*

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INTRODUCTION.

The purpose of this paper is to give simple proofs of several theorems concerning Diophantine equations of specified form and to derive certain interesting consequences of them. All of these proofs are supposed to be novel, in whole or in part; and the results in VI, VII and VIII are believed to be new.

In §1 it is shown that the Diophantine system $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$ is without a solution (in positive integers). Several of the remaining theorems of the paper are proved by means of this one. In §2 it is shown that no numerical right triangle has a square area. In §3 the impossibility (in positive integers) of the equation $p^4 - q^4 = \alpha^2$ is proved, and several consequences of this fact are obtained. In §4 I show that the equations $m^4 - 4n^4 = \pm t^2$ are both impossible (in positive integers), and derive various consequences.

Incidentally, several proofs are given of the impossibility of the Fermat equation $x^4 + y^4 = z^4$.

The principal arguments in §§1 and 4 are made by means of Fermat's famous method of "infinite descent."

§1. *Impossibility of the simultaneous equations* $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$.

In this section we shall prove the following theorem:

THEOREM I. *There do not exist integers* m, n, p, q, *all different from zero, such that*

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